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BEST APPROXIMATIONS IN PREDUALS OF VON NEUMANN ALGEBRAS

ALVARO ARIAS AND VANIA MASCIONI

Abstract

This paper characterises the semi-Chebychev subspaces of preduals of von Neumann algebras. As an application it generalises the theorem of Doob, that says that H_0^1 has unique best approximations in $L_1(T)$, to a large class of preannihilators of natural non-selfadjoint operator algebras including the nest algebras. Then it studies the semi-Chebychev subspaces of the trace class operators and shows that the only Chebychev $*$-diagrams are 'triangular'.

1. *Introduction*

This paper characterises the semi-Chebychev subspaces of preduals of von Neumann algebras; in particular, those of the trace class operators.

As an application of this, we generalise the Theorem of Doob [3] that says that H_0^1 has unique best approximations in $L^1(T)$, to a large class of preannihilators of natural 'triangular' algebras; for example, nest algebras.

In the final section we characterise the finite codimensional weak*-closed subspaces of the trace class operators and clarify the situation for the special case of $*$-diagrams.

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2. *Preliminaries*

In this paper H denotes a Hilbert space, and $c_1(H)$ the trace class operators; that is, those $T \in B(H)$ for which $\|T\|_1 = \operatorname{tr}(|T|) < \infty$. We identify $B(H)$ with c_1^* under the trace duality; that is, for $T \in c_1$ and $S \in B(H)$, $\langle T, S\rangle = \operatorname{tr}(TS)$. We also use the fact that every compact operator has a Schauder decomposition $T = \sum_i a_i(T)\, \phi_i \otimes \psi_i$, where $\phi \otimes \psi$ is the rank-1 operator sending h to $(\phi, h)\,\psi$ and the $a_i(T)$ are the singular numbers of T. Moreover, if $T \in c_1$ then $\|T\|_1 = \sum_i a_i(T)$ (see [9]).

Let M be a von Neumann algebra, and M_* the unique isometric predual (see [11]).

If $f \in M_*$ and $b \in M$ then $bf \in M_*$, $fb \in M_*$ and $f^* \in M_*$, where these are defined by

$$bf(x) = f(xb), \qquad fb(x) = f(bx), \qquad f^*(x) = \overline{f(x^*)}.$$

Now $f \in M^*$ is positive if for every $x \in M$, $f(x^*x) \geqslant 0$. We shall use the fact that $f \geqslant 0$ if and only if $f(1) = \|f\|$. We say that $f \in M_*$ is hermitian if $f^* = f$. There is also a polar decomposition in M_*. If $f \in M_*$ we can find $u \in M$, a partial isometry, such that $uf = |f|$. For more information on von Neumann algebras see [11].

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Let X be a Banach space and $G \subset X$ a closed subspace. Then G is proximinal in X if every $x \in X$ has a best approximant from G; that is, there exists $y \in G$ such that $\|x-y\| = \mathrm{d}(x, G)$. Now G is semi-Chebychev if every $x \in X$ has at most one best approximant; and G is Chebychev if every $x \in X$ has a unique one. A fundamental reference concerning best approximations is Singer's book **[10]**.

3. *The main result*

In this section we characterise the semi-Chebychev subspaces of the preduals of von Neumann algebras; in particular, those of the trace class operators, c_1.

If $G \subset M_*$ we let

$$G^{\perp} = \{b \in M : b(h) = 0 \text{ for every } h \in G\}$$

and $G^{\perp}G = \{bh : b \in G^{\perp} \text{ and } h \in G\}$. Notice that $G^{\perp}G \subset M_*$.

THEOREM 1. *Let M be a unital von Neumann algebra and $G \subset M_*$. Then G is not semi-Chebychev if and only if there exist $h \in G$, $h \neq 0$ and $b \in G^{\perp}$, $\|b\| = 1$ satisfying*

(i) *bh is hermitian,*

(ii) *$b^*bh = h$.*

REMARK. Notice that if $M = B(H)$, condition (ii) is equivalent to b is an isometry on the range of h.

As an immediate application of Theorem 1 we obtain the following.

COROLLARY 2. *Let $G \subset M_*$ such that $G^{\perp}G$ contains no non-zero hermitian element; then G is semi-Chebychev.*

Corollary 2 was proved in **[1]** for c_1 and was used to show that the non-commutative H^1-spaces in c_1 (for example, the set of upper triangular elements of c_1) are Chebychev, just as in the commutative case **[6]**. (See Section 4 for further discussion).

The proof of Theorem 1 depends on a generalisation of the following easy and well-known lemma to the von Neumann algebra setting.

LEMMA 3. *If $T \in c_1$, $B \in B(H)$ are such that $\|B\| = 1$ and $\operatorname{tr}(BT) = \|T\|_1$ then $BT = |T|$ and $B^*|T| = T$.*

For the next lemma we assume that $M \subset B(H)$ for some Hilbert space H.

LEMMA 4. *Let M be a von Neumann algebra and let $f \in M_*$, $b \in M$, $\|b\| = 1$ be such that $bf \geqslant 0$ and $\|bf\| = \|f\|$. Then we have that $bf = |f|$ and $f = b^*|f|$.*

Proof. It follows from **[2**, Theorem 12.2.5**]** that $bf = |f|$.

Find $T \in c_1$ such that for every $x \in M$, $f(x) = \operatorname{tr}(Tx)$ and $\|T\|_1 = \|f\|$. Since $\operatorname{tr}(bT) = \|T\|_1$and $\|b\| = 1$, Lemma 3 gives us that $bT = |T|$ and $b^*|T| = T$, or $b^*bT = T$. Clearly, $b^*bf = f$.

We are now ready for the proof of Theorem 1.

Proof of Theorem 1. Assume that G is not semi-Chebychev. Then we can find $f \in M_*$, $h \in G$, $h \neq 0$ and $b \in G^\perp$, $\|b\| = 1$ such that

$$\|f\| = \mathrm{d}(f, G) = \|f+h\|,$$

and $f(b) = \|f\| = \|f+h\| = (f+h)(b)$.

Since $\|bf\| \leqslant \|f\| = f(b) = bf(1) \leqslant \|bf\|$ we have that $bf \geqslant 0$ and $\|bf\| = \|f\|$. By Lemma 4, this implies that $bf = |f|$ and $b^*|f| = f$. Similarly, $b(f+h) = |f+h|$ and $b^*|f+h| = f+h$.

Hence,

$$bh = b(f+h) - bf = |f+h| - |f|$$

is clearly hermitian and

$$b^*bh = b^*|f+h| - b^*|f| = f+h-f = h.$$

Conversely, let us assume that $h \in G$, $h \neq 0$, $b \in G^\perp$, $\|b\| = 1$ are such that bh is hermitian and $b^*bh = h$.

Find $u \in M$, $\|u\| = 1$ such that $|bh| = u(bh)$. Since bh is hermitian, we have $|bh| = (bh)u^*$. Let $f = b^*|bh|$.

Claim: $\|f\| = \mathrm{d}(f, G) = \|f+h\|$.

Since $h \neq 0$ this clearly implies that G is not semi-Chebychev. We will now check the first equality of the claim.

Clearly $\|f\| \leqslant \|bh\|$; on the other hand

$$f(b) = bf(1) = bb^*|bh|(1) = bb^*bhu^*(1) = bhu^*(1) = |bh|(1) = \|bh\|.$$

Since $b \in G^\perp$ and $\|b\| = 1$, we have that $\|f\| = \mathrm{d}(f, G)$.

For the other equality notice that

$$f+h = b^*|bh| + b^*bh = b^*[|bh| + bh].$$

Since bh is hermitian, we have that

$$|bh| + bh \geqslant 0$$

and

$$\|\,|bh| + bh\| = (|bh| + bh)(1) = |bh|(1) = \|bh\|.$$

Hence, $\|f+h\| \leqslant \|bh\|$. On the other hand,

$$(f+h)(b) = f(b) + h(b) = f(b) = \|f\| = \|bh\|.$$

Therefore, $\|f+h\| = \|f\|$.

REMARK. According to [8], a subspace Y of a Banach space X has property $(\mathscr{U})$ if any $y^* \in Y^*$ has a unique Hahn–Banach extension in X^*. Phelps proved that Y has $(\mathscr{U})$ if and only if $Y^\perp$ is semi-Chebychev in X^*. It follows that Theorem 1 can be used to study this property when M_* is a dual space. It is also clear that the result can be used to find unique weak* Hahn–Banach extensions of weak*-continuous functionals on M whenever they exist. This happens when the preannihilator is proximinal.

4. PREANNIHILATORS OF SUBALGEBRAS

Doob [3] proved that H_0^1 is a semi-Chebychev subspace of $L^1(T)$. In this section we shall show that this property is shared by a large class of preannihilators of natural non-selfadjoint subalgebras of M, including the analytic algebras (see [7]), in particular, nest algebras and nest subalgebras of von Neumann algebras.

For $\mathscr{A} \subset M$ let $\mathscr{A}^* = \{x^* : x \in \mathscr{A}\}$, where * means the adjoint operation in M; and let $\mathscr{A}_\perp = \{f \in M_* : f_{|\mathscr{A}} = 0\}$ be the preannihilator.

PROPOSITION 5. *Let M be a von Neumann algebra and $\mathscr{A} \subset M$ a weak*-closed unital subalgebra of M. Then the following are equivalent:*

(i) *$\mathscr{A}_\perp$ is semi-Chebychev;*
(ii) *$\mathscr{A}_\perp$ contains no non-zero hermitian element;*
(iii) *$\mathscr{A} + \mathscr{A}^*$ is w^*-dense in M.*

Proof. (i)$\Leftrightarrow$(ii) It is easy to check that since $\mathscr{A}$ is a unital algebra it follows that $\mathscr{A}\mathscr{A}_\perp = \mathscr{A}_\perp$. Hence, the equivalence of (i) and (ii) follows directly from Theorem 1.

(ii)$\Rightarrow$(iii) If $\mathscr{A} + \mathscr{A}^*$ is not w^*-dense then there exists a non-zero $f \in \mathscr{A}_\perp \cap (\mathscr{A}^*)_\perp$. Clearly, $f^* \in \mathscr{A}_\perp \cap (\mathscr{A}^*)_\perp$ as well. Therefore, $f \pm f^* \in \mathscr{A}_\perp$ and one of them is non-zero.

(iii)$\Rightarrow$(ii) Let $f \in \mathscr{A}_\perp$ be such that $f = f^*$. It is clear that $f \in (\mathscr{A}^*)_\perp$. Since $\mathscr{A} + \mathscr{A}^*$ is w^*-dense, we have that $f = 0$.

REMARK. Notice that Proposition 5 is interesting only in the non-selfadjoint case. If $\mathscr{A} = \mathscr{A}^*$ then $\mathscr{A}_\perp$ is semi-Chebychev if and only if $\mathscr{A} = M$.

There are many natural examples where Proposition 5 applies. For instance, since $H_0^1 = H_\perp^\infty$ and $H^\infty + \overline{H^\infty}$ is w^*-dense in the von Neumann algebra $L^\infty(T)$ we get the following.

COROLLARY 6 (Doob [3]). *H_0^1 is semi-Chebychev in $L^1(T)$.*

Other important examples are the nest algebras. By a nest of projections in H we mean any linearly ordered set $\mathscr{P}$ of orthogonal projections which is closed in the strong operator topology and contains 0 and I. The nest algebra induced by $\mathscr{P}$ is the set of all operators $T \in B(H)$ that leave invariant every element of $\mathscr{P}$; that is,

$$\operatorname{Alg}\mathscr{P} = \{T \in B(H) : (I-P)TP = 0 \text{ for every } P \in \mathscr{P}\}.$$

It is well known that $\operatorname{Alg}\mathscr{P} + (\operatorname{Alg}\mathscr{P})^*$ is w^*-dense in $B(H)$. Hence, we recover the result from [1].

COROLLARY 7 [1]. *$(\operatorname{Alg}\mathscr{P})_\perp$ is semi-Chebychev in c_1.*

This result is true for a more general type of nest algebras. If $\mathscr{P} \subset M$ one defines $\mathscr{A} = \operatorname{Alg}\mathscr{P} \bigcap M$. Nest subalgebras of von Neumann algebras have been studied by Gilfeather and Larson [4] and it is known that $\mathscr{A} + \mathscr{A}^*$ is w^*-dense in M (see [7] or [5]). Hence, the proposition applies.

And finally, if $S \in B(H)$ let $M = \{S\}''$ be the von Neumann algebra generated by S. Let

$$\mathscr{A}_S = \overline{\{p(S) : p \text{ is a polynomial}\}}^{w^*}.$$

It is easy to see that $\mathscr{A}_S$ satisfies property (iii).

Since $\mathscr{A}_\perp$ is semi-Chebychev in M_* one could believe that $\mathscr{A}^\perp$, the annihilator of $\mathscr{A}$ in M^*, is semi-Chebychev in M^*. This would imply that any continuous linear functional on $\mathscr{A}$ has a unique Hahn–Banach extension to M. Unfortunately this is not the case for our natural candidates.

PROPOSITION 8. *Let $\mathscr{B}$ be a unital* C*-*algebra and $\mathscr{A} \subseteq M$ a unital subalgebra. Then the following are equivalent:*

(i) *$\mathscr{A}^{\perp}$ is Chebychev;*

(ii) *$\mathscr{A}+\mathscr{A}^*$ is norm dense in $\mathscr{B}$.*

Proof. Since $\mathscr{A}^{\perp}$ is w^*-closed and $\mathscr{B}^*$ is a dual space, it follows that $\mathscr{A}^{\perp}$ is proximinal. Notice that $\mathscr{B}^*$ is the predual of the von Neumann algebra $\mathscr{B}^{**}$. We are going to use Theorem 1 for $\mathscr{A}^{\perp}$ and $\mathscr{A}^{\perp\perp}$. Notice also that $\mathscr{A}$ is w^*-dense in $\mathscr{A}^{\perp\perp}$.

(i)$\Rightarrow$(ii) If $\mathscr{A}+\mathscr{A}^*$ is not norm dense in $\mathscr{B}$ we can find $f\in\mathscr{B}^*\backslash\{0\}$ such that $f\in\mathscr{A}^{\perp}\cap(\mathscr{A}^*)^{\perp}$. Since $f^*\in\mathscr{A}^{\perp}\cap(\mathscr{A}^*)^{\perp}$ we can assume that $f=f^*$. The proof follows from Theorem 1 by noticing that $1\in\mathscr{A}^{\perp\perp}$, $1f$ is hermitian and $1^*1f=f$.

(ii)$\Rightarrow$(i) If $\mathscr{A}^{\perp}$ is not semi-Chebychev in $\mathscr{B}^*$ we can find $h\in\mathscr{A}^{\perp}\backslash\{0\}$ and $b\in\mathscr{A}^{\perp\perp}$, $\|b\|=1$ such that bh is hermitian and $b^*bh=h$. It is easy to see that $bh\in\mathscr{A}^{\perp}$. Since bh is hermitian it follows that $bh\in\mathscr{A}^{\perp}\cap(\mathscr{A}^*)^{\perp}$.

REMARK. It is well known that nest algebras in $B(H)$ do not satisfy condition (ii). This implies that biduals of nest algebras are not nest subalgebras of the von Neumann algebra $B(H)^{**}$ and also provides an example of a Chebychev subspace $X\subset Y$ such that X^{**} is not Chebychev in Y^{**}.

In contrast, if $\mathscr{B}$ is an AF-algebra and $\mathscr{A}$ is a strongly maximal triangular subalgebra of $\mathscr{B}$, then $\mathscr{A}$ satisfies (ii) (see [**12**]). Hence, $\mathscr{A}^{\perp}$ is Chebychev in $\mathscr{B}^*$. As an immediate corollary we obtain the following.

COROLLARY 9. *Let $\mathscr{A}$ be a strongly maximal triangular subalgebra of the* AF-*algebra $\mathscr{B}$. Then the Hahn–Banach extensions on $\mathscr{A}$ to $\mathscr{B}$ are unique.*

Proposition 8 and the analogue of Corollary 9 apply also for $\mathscr{B}=C(T)$, the space of continuous functions and on the unit circle, and $\mathscr{A}=A$, the disk algebra.

5. *Semi-Chebychev subspaces of the trace class operators*

In this section we study some of the natural semi-Chebychev subspaces of the trace class operators. We start with the weak*-closed n-codimensional subspaces.

PROPOSITION 10. *Let $G\subset c_1$ be a weak*-closed, n-codimensional subspace. Let*

$$G^{\perp}=[R_1,\dots,R_n]\subset K(H).$$

Then the following are equivalent.

(i) *G is Chebychev.*

(ii) *Whenever $R\in G^{\perp}$, $\|R\|=1$, $R=\sum_i a_i(R)\,\phi_i\otimes\psi_i$ (with $(\phi_i),(\psi_i)$ orthonormal) we have*

(a) $m=\max\{i\colon a_i(R)=1\}\leqslant n$,

(b) $\operatorname{rank}[(R\phi_i,R_j\phi_i)]_{i=1,\dots,m,j=1,\dots,n}=m$.

Proof. (i)$\Rightarrow$(ii) Suppose that $R\in G^{\perp}$, $\|R\|=1$, $R=\sum_i a_i(R)\,\phi_i\otimes\psi_i$, where (ϕ_i), (ψ_i) are orthonormal, and let $m=\max\{i\colon a_i(R)=1\}$. If R fails (a), that is, if $m>n$, consider the system of linear equations

$$\sum_{i=1}^{n+1}\alpha_i(R_j\phi_i,\psi_i)=0,\qquad j=1,\dots,n.$$

Clearly, there is a non-trivial solution $(\alpha_1, \ldots, \alpha_{n+1})$. Define

$$A \equiv \sum_{i=1}^{n+1} \alpha_i \psi_i \otimes \phi_i$$

and note that $A \in G \backslash \{0\}$. Further, R is isometric on $A(H) \subset [\phi_1, \ldots, \phi_{n+1}]$ (since $m > n$) and $RA = \sum_i \alpha_i \psi_i \otimes \psi_i$ is selfadjoint. By Theorem 1, G is not semi-Chebychev.

If R fails (b) but not (a), the system

$$\sum_{i=1}^{m} \alpha_i (R\phi_i, R_j \phi_i) = 0, \qquad j = 1, \ldots, n$$

has a non-trivial solution $(\alpha_1, \ldots, \alpha_m)$. In this case $A = \sum_i \alpha_i \psi_i \otimes \phi_i$ will do the trick as above.

(ii) $\Rightarrow$ (i) Since G is weak*-closed, if G is assumed to be not Chebychev, it is not even semi-Chebychev, and so Theorem 1 states that there are $A \in G \backslash \{0\}$, $R \in G^\perp$ such that $\|R\| = 1$, R is isometric on the range of A and RA is selfadjoint. So, if rank $A > n$, we must have $m \equiv \max\{i : a_i(R) = 1\} > n$ and R fails (a). If rank $A = k \leqslant m \leqslant n$, let

$$A = \sum_{i=1}^{k} \alpha_i \psi_i \otimes \phi_i, \qquad R = \sum_{i=1}^{\infty} a_i(R) \phi_i \otimes \psi_i$$

with some orthonormal sequences $(\phi_i), (\psi_i)$, and consider that since the system

$$\sum_{i=1}^{k} \alpha_i (R\phi_i, R_j \phi_i) = 0, \qquad j = 1, \ldots, n$$

has the non-trivial solution $(\alpha_1, \ldots, \alpha_k)$, R must fail (b).

Remark. It is interesting to note the analogy between Proposition 10 and the similar statement for l_1 (see [**10**, III.2.11]): let $G \subset l_1$ be weak*-closed and n-codimensional; that is, $G^\perp = [\beta_1, \ldots, \beta_n]$ and $\beta_i \in c_0$. Then G is Chebychev if and only if for any $\beta \in G^\perp$ such that $\beta(k) = \|\beta\|$ for some k we have

$$m \equiv \operatorname{card}\{k : |\beta(k)| = \|\beta\|\} \leqslant n$$

and, if $\{k_1, \ldots, k_m\}$ is the set of coordinates where β attains its norm,

$$\operatorname{rank}[\beta_j(k_i)]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = m.$$

We have also some results about the finite dimensional subspaces.

Proposition 11. *If $G \subset c_1$ and G is not semi-Chebychev, then it contains a non-zero operator A such that $\sum_i \varepsilon_i a_i(A) = 0$ for an appropriate choice of $\varepsilon_i \in \{-1, 1\}$. Further, if G is one-dimensional, the converse holds, too.*

Proof. If G is not semi-Chebychev we can find $U \in G^\perp$, $\|U\| = 1$ and $A \in G \backslash \{0\}$ such that UA is selfadjoint and U is an isometry on the range of A. We see that, since

$$A = \sum_i \lambda_i \phi_i \otimes \psi_i$$

with $(\phi_i), (\psi_i)$ orthonormal (we keep the same notation), then $a_i(A) = |\lambda_i|$ for all i and so, defining $\varepsilon_i \equiv \operatorname{sign} \lambda_i$, we get

$$\sum_i \varepsilon_i a_i(A) = \sum_i \lambda_i = \operatorname{tr} UA = 0.$$

If G is one-dimensional, let (ε_ι) be such that $\varepsilon_\iota \in \{-1, 1\}$ for all i and $\sum_\iota \varepsilon_\iota a_\iota(A) = 0$ for some $A \in G$. Let $A = \sum_\iota a_\iota(A)\, \phi_\iota \otimes \psi_\iota$ for some orthonormal sequences $(\phi_\iota), (\psi_\iota)$, and define U by $U\psi_\iota \equiv \varepsilon_\iota \phi_\iota$. We see that $U \in G^\perp$, $\|U\| = 1$, U is isometric on the range of A and UA is selfadjoint. By Theorem 1, G is not semi-Chebychev.

In [6], Kahane studied the approximation properties of the following subspaces of $L^1(T)$. If $\Lambda \subseteq Z$ let $L^1_\Lambda(T) = \{f \in L^1 : \hat{f}(n) = 0 \text{ if } n \notin \Lambda\}$. If $\Lambda = Z^+$ then $L^1_\Lambda = H^1_0$. He proved that L^1_Λ is semi-Chebychev if and only if $\Lambda = (2p-1)Z$ or $\Lambda = (2p-1)Z^+$, where $p \in Z$.

The analogue of these examples in c_1 are the $*$-diagrams.

Let $(e_\iota)_{\iota \in I}$ be a fixed orthonormal basis of H, and let $\Lambda \subset I \times I$. By the $*$-diagram induced by Λ we mean the class of all operators T in c_1 which have no component outside Λ; that is, such that $(\iota, \kappa) \notin \Lambda$ implies that $(e_\iota, Te_\kappa) = 0$. Since $*$-diagrams are clearly weak*-closed subspaces of c_1 they are proximinal [10, I.2.5]. In [1] it has been shown that the converse of Corollary 2 holds for $*$-diagrams; that is, *a $*$-diagram G is Chebychev if and only if $G^\perp G$ does not contain non-zero self-adjoint elements.* Actually, it was asked whether this is true in arbitrary weak*-closed subspaces of c_1. In this section we answer that question in the negative and show that the Chebychev $*$-diagrams are in some sense triangular. This explains why the result is true for them, keeping in mind the fact that T_1 (the subspaces of the operators having an upper triangular matrix) is Chebychev in c_1 (see [1]).

COUNTEREXAMPLE. Fix $\iota \neq \kappa \in I$ and let

$$G \equiv \{T \in c_1 : (e_\iota, Te_\iota) + \tfrac{1}{2}(e_\kappa, Te_\kappa) = 0\}.$$

Then G is Chebychev and $G^\perp G$ contains a non-zero selfadjoint operator.

To see this note first that $G^\perp = [U]$ for $U = e_\iota \otimes e_\iota + \frac{1}{2} e_\kappa \otimes e_\kappa$. Since

$$a_2(U) = \tfrac{1}{2} < a_1(U) = 1,$$

Proposition 10 easily implies that G is Chebychev. On the other hand, taking $A = e_\iota \otimes e_\iota - 2e_\kappa \otimes e_\kappa$, we have $A \in G \backslash \{0\}$ and UA is clearly selfadjoint.

We now prove that the Chebychev $*$-diagrams are 'triangular'.

THEOREM 12. *Let H have the orthonormal basis $(e_\iota)_{\iota \in I}$ (I a totally ordered set) and let $\mathcal{G} \subset c_1(H)$ be a Chebychev $*$-diagram. We can find $D, \tilde{D}$, totally ordered sets, having the same cardinality as I and unitaries $U : l_2(D) \to H$, $V : H \to l_2(\tilde{D})$ such that $\tilde{\mathcal{G}} = V\mathcal{G}U \subset c_1(l_2(D), l_2(\tilde{D}))$ is a $*$-diagram which is 'triangular' with respect to the natural bases of $l_2(D)$ and $l_2(\tilde{D})$; that is, if we have a zero at $(d, \tilde{d})$ then we have a zero at $(d', \tilde{d})$ for all $d' < d$ and at $(d, \tilde{d}')$ for all $\tilde{d}' > \tilde{d}$; and if we have a $*$ at $(d, \tilde{d})$, then we have a $*$* at $(d', \tilde{d})$ for $d' > d$ *and at $(d, \tilde{d}')$ for $\tilde{d}' < \tilde{d}$.*

The proof depends on the following lemma that appears in [1].

LEMMA 13. *The $*$-diagram $\mathcal{G} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ is not Chebychev.*

As remarked in [1], note that Lemma 13 can easily be generalised. If we have a general $*$-diagram such that, for some rows $\iota_1 \neq \iota_2$ and columns $\kappa_1 \neq \kappa_2$ we have a $*$ both at (ι_1, κ_1) and at (ι_2, κ_2), and a zero both at (ι_1, κ_2) and at (ι_2, κ_1), then $\mathscr{G}$ is not Chebychev.

Proof of Theorem 12. Let $\mathscr{G}$ be a Chebychev $*$-diagram. Define new order relations $\leqslant_r$, $\leqslant_c$ between the 'rows' and 'columns' of $\mathscr{G}$ in the following way:

$$\operatorname{row} \iota_1 \leqslant_r \operatorname{row} \iota_2$$

if, whenever we have a zero at (ι_1, κ) in $\mathscr{G}$, we have a zero at (ι_2, κ) in $\mathscr{G}$. Proceed similarly with the columns to define $\leqslant_c$. It follows from the remark after Lemma 13 that any two rows or columns of $\mathscr{G}$ are comparable (otherwise we could reproduce the pattern of Lemma 13) and so $D = (I, \leqslant_r)$ and $\tilde{D} = (I, \leqslant_c)$ are totally ordered sets having the same cardinality as I. The isometries $U: l_2(D) \to H$ and $V: H \to l_2(\tilde{D})$ defined by the 'identities' $D \to I$ and $I \to D$ can be regarded as an operation of changing rows and columns in the 'matrix' of $\mathscr{G}$. It is now just a matter of time to verify that the 'matrix' of $V\mathscr{G}U \subset c_1(l_2(D), l_2(\tilde{D}))$ is triangular in the sense described above.

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